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On the 2N-Widths of a Periodic Sobolev Class*

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Let $Q(x)$ be a polynomial with real coefficients and $W_p(Q(D))$ be the periodic Sobolev class defined by $Q(D)$, $D = d/dt$. We get the exact values of Kolmogorov, Gel'fand, and linear $2n$-widths of $W_p(Q(D))$ in L_p for $p \in (1, \infty)$ and $n > N(Q)$, where $N(Q)$ is a constant dependent on Q. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let $Q(x)$ be a polynomial with real coefficients and the Sobolev class $W_p(Q(D))$ be the class of continuous 2π-periodic functions $f(x)$ for which $f^{(\deg Q - 1)}$ is absolutely continous and $\|Q(D) f\|_p \leqslant 1$, where $\deg Q$ is the degree of Q, $D = d/dt$, and $\|\cdot\|_p$ is the usual $L_p[0, 2\pi]$-norm. Denote by $d_n(p, q)$, $d^n(p, q)$, $\delta_n(p, q)$, and $b_n(p, q)$ the Kolmogorov, Gel'fand, linear, and Bernstein n-widths of $W_p(Q(D))$ in $L_q[0, 2\pi]$, respectively. When $Q(x)$ has only real zeros, the quantities $s_n(p, q)$ have long been investigated by many authors (cf. [1–3, 12, 14]), where s_n denotes any of the four symbols d_n, d^n, δ_n, and b_n. In the case where $Q(x)$ has complex zeros, the Bernoulli function

$$G(x) = \frac{1}{2\pi} \sum_{Q(im) \neq 0} \frac{e^{imx}}{Q(im)}, \qquad i = \sqrt{-1}, \tag{1.1}$$

corresponding to $Q(D)$ does not satisfy the property of cyclic variation-diminishing. Therefore the study of $s_n(p, q)$ becomes complicated. This question has been discussed in several papers up to now (cf. [4, 5, 9–11, 16]). In this paper, we obtain the exact values of $d_{2n}(p, p)$, $d^{2n}(p, p)$, and $\delta_{2n}(p, p)$ for $p \in (1, \infty)$ and $n > N(Q)$, where $N(Q)$ is a constant determined simply by Q.

Similar results about n-widths of this paper have been proved by Pinkus [13] for nonperiodic Sobolev classes. By discretization, [13] proves, for

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$p = q$, the existence of the function and the eigenvalue satisfying the critical point equation [13, Thm. 2.1]. This is essential to the estimations of n-widths. The proof of Theorem 2.1 in [13] is complicated and does not adapt to the periodic case. We prove, by a variational condition, for $p, q \in (1, \infty)$, the existence of function and eigenvalue satisfying the critical point equation. Moreover, the expression for the eigenvalue has been given. Our method is different from that of [1] as well.

Throughout the paper G is given by (1.1).

2. PRELIMINARY FACTS

Set $Q(x) = \prod_{j=1}^{r} (x - \lambda_j) \prod_{j=1}^{\mu(Q)} [(x - a_j)^2 + b_j^2]$, where $r + 2\mu(Q) = \deg Q$, $\lambda_j, a_j \in \mathbf{R}$, and $b_j > 0$. Denote by $\nu(Q)$ the maximum of the b_j, $j = 1, ..., \mu(Q)$. We define

$$E(G) = \operatorname{span}\{\cos mx, \sin mx \mid Q(im) = 0\},$$

and denote by $E^{\perp}(G)$ the orthogonal complement of $E(G)$ in $L_p[0, 2\pi]$.

It is well known (cf. [9, p. 1360]) that $f \in W_p(Q(D))$ if and only if f may be represented as

$$f(x) = P(x) + (G * h)(x),$$

where $P \in E(G)$, $h \in E^{\perp}(G)$, $h = Q(D) f$ a.e., and

$$(G * h)(x) = \int_0^{2\pi} G(x - t)\, h(t)\, dt.$$

Let f be a 2π-periodic function. Denote by $Z(f)$ the number of zeros of f on a period, counting multiplicities, and by $Z_s(f)$ the number of zeros of f on a period, counting multiplicities up to s. By $\operatorname{dis}(f)$ we denote the maximum distance between consecutive zeros of f, in which a zero interval is regarded as a zero point.

If $\nu(Q) = 0$, i.e., $Q(x)$ has only real zeros, all the results of this paper were established in [2]. So we assume $\nu(Q) > 0$.

LEMMA 2.1 [9, pp. 1357, 1360]. *If* $\operatorname{dis}(f) < \pi/(2\mu(Q) - 1)\,\nu(Q)$, *then*

(1) $Z_m(f) \leqslant Z_{m - \deg Q}(Q(D) f)$, $f \in C^m$, $m \geqslant \deg Q$;

(2) $S_c(f) \leqslant S_c(Q(D) f)$, *for* $f(x) \in W_p(Q(D))$, *where* $S_c(f)$ *is the number of sign changes of periodic* f (*cf.* [12, p. 60]).

Let $T = \{x_i\}_{i=1}^{m} \subseteq [0, 2\pi)$. If $x_1 < \cdots < x_m$, we set

$$d(T) = \max_{1 \leqslant i \leqslant m} (x_{i+1} - x_i), \qquad x_{m+1} = 2\pi + x_1.$$

The G-spline subspace $X(T)$, with simple knots $\{x_i\}_{i=1}^m$, is the class of functions

$$F(x) = P(x) + \sum_{i=1}^{m} c_i G(x - x_i), \qquad P \in E(G), \qquad \sum_{i=1}^{m} c_i\, \delta(x - x_i) \in E^{\perp}(G),$$

where $\delta(x)$ is 2π-periodic Dirac function.

In what follows we need to smooth functions. This is done by means of convoluting f with

$$\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{k=-\infty}^{\infty} \exp\left(-\left(\frac{x - 2k\pi}{\sqrt{2}\,\sigma}\right)^2\right), \qquad \sigma > 0.$$

Let G be given in (1.1). We set $G_\sigma = \phi_\sigma * G$ and call $X(T, \sigma) := \{\phi_\sigma * F \mid F \in X(T)\}$ the G_σ-spline subspace with simple knots at T.

It is easy to see that $s \in X(T, \sigma)$ if and only if s has the representation $s(x) = P(x) + \sum_{i=1}^{m} c_i G_\sigma(x - x_i)$, where $P(x)$ and c_i's satisfy the same conditions as in the definition of $X(T)$.

LEMMA 2.2 [6, p. 457]. (1) *For any 2π-periodic function $g \in L_1[0, 2\pi]$ we have*

$$Z(\phi_\sigma * g) \leqslant S_c(g).$$

(2) *If g is continuous with period 2π, then*

$$\lim_{\sigma \to 0^+} \|\phi_\sigma * g - g\|_c = 0.$$

LEMMA 2.3 [9, Lemma 3.4]. *Let $m > 0$. If a nontrivial $F \in X(T, \sigma)$ satisfies $dis(F) < \pi/(2\mu(Q) - 1)\, v(Q)$, then*

$$Z(F) \leqslant \text{card } T.$$

LEMMA 2.4. *Let card $T = 2m + 1$ and $F \in X(T, \sigma)$ be nontrivial. If F vanishes on $T' = \{y_j\}_{j=1}^{2m} \subseteq [0, 2\pi)$ with $d(T') < \pi/(2\mu(Q) - 1)\, v(Q)$, then $S_c(F) = 2m$ and F changes sign at the y_j, $j = 1, 2, ..., 2m$.*

Proof. We need only to prove that for any $j \in \{1, ..., 2m\}$, y_j is a simple zero of F and that F has no zero except $\{y_j\}_{j=1}^{2m}$.

By Lemma 2.3, F has no interval zero and $Z_3(F) \leqslant 2m + 1$. Therefore, $2m \leqslant Z_3(F) \leqslant 2m + 1$. The proof will be complete if $Z_3(F) = 2m$. Assume to the contrary that $Z_3(F) = 2m + 1$. Then there are two cases as follows.

(1) There exists a $k \in \{1, ..., 2m\}$ such that $0 = F(y_k) = F'(y_k) \neq F''(y_k)$ and $0 = F(y_j) \neq F'(y_j)$ for $j \in \{1, ..., 2m\}/\{k\}$. Therefore F has no

zero except T'. From these it follows that F only changes sign at $T'/\{y_k\}$ and thus that $S_c(F)=2m-1$.

(2) For any $y_j \in T'$, $0=F(y_j)\neq F'(y_j)$. Then there exists a $x_0\in[0,2\pi)/T'$ such that $0=F(x_0)\neq F'(x_0)$. Therefore, $S_c(F)=2m+1$.

In all cases $S_c(F)$ is an odd number. This contradicts the fact that $S_c(F)$ is even. Therefore the proof is complete.

Remark. Given Q, we call $Q^*(D)=Q(-D)$ the conjugate operator of $Q(D)$. Obviously the Bernoulli function corresponding to $Q^*(D)$ is $G^*(x)=G(-x)$. Therefore, $E(G^*)=E(G)$, $\mu(Q^*)=\mu(Q)$ and $\nu(Q^*)=\nu(Q)$. Any result established for $W_p(Q(D))$ has its analogue for $W_p(Q^*(D))$. For example, let $f(x)=P(x)+(G^* * h)(x)\in W_p(Q^*(D))$ with $\operatorname{dis}(f)<\pi/(2\mu(Q)-1)\,\nu(Q)$, then $S_c(f)\leqslant S_c(h)$.

We introduce for convenience the following notation:

$$M\begin{pmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m\end{pmatrix} = \begin{vmatrix} 0 & \cdots & 0 & g_1(y_1) & \cdots & g_1(y_m) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & g_\nu(y_1) & \cdots & g_\nu(y_m) \\ g_1(x_1) & \cdots & g_\nu(x_1) & G(x_1-y_1) & \cdots & G(x_1-y_m) \\ \vdots & & \vdots & \vdots & & \vdots \\ g_1(x_m) & \cdots & g_\nu(x_m) & G(x_m-y_1) & \cdots & G(x_m-y_m)\end{vmatrix},$$

where $g_1, ..., g_\nu$ is a basis of $E(G)$. The determinant will be denoted by $M_\sigma\left(\begin{smallmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m\end{smallmatrix}\right)$ if G is replaced by G_σ.

3. Estimation of $\delta_{2n}(p,q)$ $(1<q\leqslant p<\infty)$ from Above

For $p,q\in(1,\infty)$, $n=1,2,...$, we consider the following extremal problems:

$$\lambda_n(p,q,G):=\sup\{\|G*h\|_q \mid h\in D_{n,p}\},$$

where $D_{n,p}$ is the class of functions $h(x)$ such that $\|h\|_p\leqslant 1$ and

$$h\left(x+\frac{\pi}{n}\right)=-h(x), \tag{3.1a}$$

$$h(x)\geqslant 0, \qquad x\in\left[0,\frac{\pi}{n}\right). \tag{3.1b}$$

Obviously, $D_{n,p} \subseteq E^{\perp}(G)$ for $n > \nu(Q)$. We will denote $\lambda_n(p, q, G)$ by λ_n if no confusion arises.

LEMMA 3.1. *$\lambda_n > 0$ for $n > \nu(Q)$.*

Proof. If $\lambda_n = 0$ for some $n > \nu(Q)$, we can choose a $\hat{h} \in D_{n,p}$ such that $\hat{h}(x) = c > 0$ on $[0, \pi/n)$. Therefore $G * h = 0$. By differentiating, we get

$$\sum_{k=0}^{2n-1} (-1)^k G\left(x - \frac{k\pi}{n}\right) = 0,$$

which contradicts the fact that $\dim X(T_n) = \operatorname{card} T_n = 2n$, where $T_n = \{k\pi/n\}_{k=0}^{2n-1}$. The proof is complete.

THEOREM 3.1. *For $n > (2\mu(Q) - 1)\,\nu(Q)$ and $1 < p, q < \infty$, there exists a unique continuous function $h_n \in D_{n,p}$ such that*

(1) $\|G * h_n\|_q = \lambda_n := \lambda_n(p, q, G) = \lambda_n \|h_n\|_p$;

(2) $\int_0^{2\pi} G(x-y)\, |(G * h_n)(x)|^{q-1} \operatorname{sgn}[(G * h_n)(x)]\, dx = \lambda_n^q\, |h_n(y)|^{p-1} \operatorname{sgn}[h_n(y)]$, $\forall y \in [0, 2\pi]$;

(3) $\operatorname{sgn}(G * h_n)(x) = \varepsilon \operatorname{sgn} \sin n(x - \beta)$, $x \in [0, 2\pi)$, *where* $\beta \in [0, \pi/n)$ *fixed*;

(4) $|h_n(x)|^{p-1} \operatorname{sgn} h_n(x)$ *has exactly* $2n$ *zeros* $\{k\pi/n\}_{k=0}^{2n-1} \subseteq [0, 2\pi)$, *all of which are simple.*

Proof. Our proof follows the same lines as that of Theorem 2.1 found in [2]. But the method needs some improvement. If a continuous function $h_n \in D_{n,p}$ satisfies (1) and (2), in view of Lemma 2.1, $\operatorname{dis}(h_n) \leqslant \pi/n$, $\operatorname{dis}(G * h_n) \leqslant \pi/n$, we can prove (3), (4), and the unicity of h_n by applying the same methods as that found in [13] (Proposition 2.8 and Proposition 2.3, respectively) and Rolle's Theorem (cf. [9, Prop. 1.3]). Thus we only prove (1) and (2). By using the weak* compactness of L_p, we can prove that there exists an $h_n \in D_{n,p}$ such that $\|G * h_n\|_q = \lambda_n = \lambda_n \|h_n\|_p$. (The procedure of the proof is the same as in [7]). For an arbitrary $h \in D_{n,p}$, $t \geqslant 0$, set

$$g(t) := \frac{\|G * (h_n + th)\|_q}{\|h_n + th\|_p}.$$

From the inequality $g(t) \leqslant g(0)$ it follows that $g'(0^+) \leqslant 0$, i.e.,

$$\int_0^{2\pi} (G * h)(x)\, |(G * h_n)(x)|^{q-1} \operatorname{sgn}[(G * h_n)(x)]\, dx$$

$$- \lambda_n^q \int_0^{2\pi} h(y)\, |h_n(y)|^{p-1} \operatorname{sgn}[h_n(y)]\, dy \leqslant 0.$$

Substituting $\int_0^{2\pi} G(x-y)\,h(y)\,dy$ for $(G*h)(x)$, and changing the order of integration, we get

$$\int_0^{2\pi} h(y)\left\{\int_0^{2\pi} G(x-y)\,|(G*h_n)(x)|^{q-1}\operatorname{sgn}[(G*h_n)(x)]\,dx - \lambda_n^q\,|h_n(y)|^{p-1}\operatorname{sgn} h_n(y)\right\}dy \leqslant 0. \tag{3.2}$$

Denote the function in $\{\cdots\}$ by $E_n(y)$. Then $E_n(y)$ satisfies (3.1a). Therefore $h(y)\,E_n(y)$ is a function with period π/n. Thus (3.2) is equivalent to

$$\int_0^{\pi/n} E_n(y)\,h(y)\,dy \leqslant 0 \qquad \text{for all} \quad h \in D_{n,p}.$$

From the arbitrariness of $h \in D_{n,p}$, we get

$$E_n(y) \leqslant 0 \qquad \text{a.e.} \quad y \in \left[0, \frac{\pi}{n}\right]. \tag{3.3}$$

If we speciafically take $h = h_n$, then $g'(0^+) = 0$, i.e.,

$$\int_0^{\pi/n} E_n(y)\,h_n(y)\,dy = 0.$$

Since $E_n(y)\,h_n(y) \leqslant 0$, a.e. $y \in [0, \pi/n]$, we get

$$E_n(y) = 0, \qquad \text{a.e.} \quad y \in F_+ := \left\{y \mid y \in \left[0, \frac{\pi}{n}\right], h_n(y) > 0\right\}. \tag{3.4}$$

Write

$$H_n(y) = \int_0^{2\pi} G(x-y)\,|(G*h_n)(x)|^{q-1}\operatorname{sgn}[(G*h_n)(x)]\,dx.$$

Then $H_n(y)$ satisfies (3.1a). So $\operatorname{dis}(H_n) \leqslant \pi/n < \pi/(2\mu(Q)-1)\,v(Q)$. It is easy to prove that

$$|(G*h_n)(x)|^{q-1}\operatorname{sgn}[(G*h_n)(x)] \in E^{\perp}(G^*) = E^{\perp}(G).$$

From Lemma 2.1 and the remark in Section 2, we obtain

$$S_c(H_n) \leqslant S_c(|G*h_n|^{q-1}\operatorname{sgn}[G*h_n]) = S_c(G*h_n).$$

Since $\operatorname{dis}(G*h_n) \leqslant \pi/n$, $h_n \in E^{\perp}(G)$, we have $S_c(G*h_n) \leqslant S_c(h_n) = 2n$. It follows that $S_c(H_n) = 2n$. We claim that the following are equivalent:

(i) $H_n(y) \geqslant 0$, $\forall y \in [0, \pi/n]$,

(ii) $E_n(y) = 0$, a.e. $y \in [0, \pi/n]$.

In fact, suppose that (i) holds. Set $F_0 = \{y \mid y \in [0, \pi/n], h_n(y) = 0\}$. From (3.3), (i) and the equality

$$H_n(y) = E_n(y) + \lambda_n^q |h_n(y)|^{p-1} \operatorname{sgn} h_n(y), \qquad y \in \left[0, \frac{\pi}{n}\right], \tag{3.5}$$

it follows that $0 \leqslant H_n(y) = E_n(y) \leqslant 0$, a.e. $y \in F_0$. Hence $H_n(y) = E_n(y) = 0$ a.e. $y \in F_0$. From this and (3.4), we get (ii).

Conversely, assume (ii) holds. If (i) is false, then by the continuity of $H_n(y)$, there exists a nondegenerate interval $I \subseteq [0, \pi/n]$ such that $H_n(y) < 0$, $\forall y \in I$. From (3.4) and (3.5), it follows that $h_n(y) = 0$, a.e. $y \in I$. Therefore, again by (3.5), we get $E_n(y) = H_n(y) < 0$, a.e. $y \in I$. This contradicts (ii). So we have proved the equivalence of (i) and (ii).

Now we are going to prove (i). We again use the method of proof by contradiction. Suppose that (i) is false. We may assume, without loss of generality, there exists $\alpha \in (0, \pi/n)$ such that

$$H_n(y) \geqslant 0 \qquad \text{for all} \quad y \in [0, \alpha);$$

$$H_n(y) \leqslant 0 \qquad \text{for all} \quad y \in \left[\alpha, \frac{\pi}{n}\right);$$

and any of the inequalities can't become equality for all y. From (3.4) and (3.5) it follows that $h_n(y) = 0$ a.e. $y \in [\alpha, \pi/n]$. Let's so modify the definition of $h_n(y)$ that it equals to zero for all $y \in [\alpha, \pi/n]$. Put $h_n^*(y) = h_n(y - \pi/n + \alpha)$, $E_n^*(y) = E_n(y - \pi/n + \alpha)$ and $H_n^*(y) = H_n(y - \pi/n + \alpha)$. It's obvious that $h_n^* \in D_{n,p}$, and $\|G * h_n^*\| = \lambda_n$. We can make the same argument for h_n^*, E_n^* and H_n^* as we have done for h_n, E_n, and H_n. Therefore the following are also equivalent:

(i)* $H_n^*(y) \geqslant 0$, $y \in [0, \pi/n)$,

(ii)* $E_n^*(y) = 0$ a.e. $y \in [0, \pi/n)$.

Obviously, $H_n^*(y) \geqslant 0$ for $y \in [0, \pi/n]$ (notice that $S_c(H_n^*) = 2n$, and $H_n(y) \geqslant 0$ for $y \in [\alpha - (\pi/n), \alpha]$).

Therefore $E_n^*(y) = 0$ a.e. $y \in [0, \pi/n]$, which entails the validity of (i)* and (ii)*. Since both $E_n(x)$ and $E_n^*(x)$ satisfy (3.1a), we have proved the validity of (ii) by the relation between $E_n(x)$ and $E_n^*(x)$. Consequently, (2) holds for almost all $y \in [0, 2\pi]$. Let us modify the definition of $h_n(x)$ in some zero-measure set, such that equality (2) holds everywhere. So h_n is continuous. The proof is complete.

Substituting $G_\sigma = \phi_\sigma * G$ for G we define $\lambda_{n,\sigma} = \lambda_n(p, q, G_\sigma)$. For G_σ, Theorem 3.1 holds as well. Denote by $h_{n,\sigma}$ the unique function satisfying Theorem 3.1 (corresponding to G_σ).

LEMMA 3.2. *For $n > (2\mu(Q) - 1)\, \nu(Q)$,*

(1) $\lim_{\sigma \to 0^+} \lambda_{n,\sigma} = \lambda_n$.

(2) *There exists a sequence of positive numbers $\{\sigma_k\}_{k=1}^{\infty}$, which converges to zero, and the corresponding sequence of continuous functions $\{h_{n,\sigma_k}\}_{k=1}^{\infty}$ converges uniformly to h_n, where h_n is given in Theorem 3.1.*

Proof. Since $\|\phi_\sigma * h_{n,\sigma}\|_p \leqslant \|\phi_\sigma\|_1 \cdot \|h_{n,\sigma}\|_p \leqslant 1$, $S_c(\phi_\sigma * h_{n,\sigma}) = 2n$ and $(\phi_\sigma * h_{n,\sigma})(x + (\pi/n)) = -(\phi_\sigma * h_{n,\sigma})(x)$, there exists an $\varepsilon_\sigma = +1$ or -1, a $\alpha_\sigma \in [0, \pi/n)$ such that

$$f_{n,\sigma} := \varepsilon_\sigma(\phi_\sigma * h_{n,\sigma})(x + \alpha_\sigma) \in D_n.$$

Therefore,

$$\lambda_n \geqslant \frac{\|G * f_{n,\sigma}\|_q}{\|f_{n,\sigma}\|_p} = \frac{\|G * (\phi_\sigma * h_{n,\sigma})\|_q}{\|\phi_\sigma * h_{n,\sigma}\|_p}$$
$$\geqslant \|G_\sigma * h_{n,\sigma}\|_q = \lambda_{n,\sigma}.$$

On the other hand, $\lambda_{n,\sigma} \geqslant \|G_\sigma * h_n\|_q = \|\phi_\sigma * (G * h_n)\|_q \to \lambda_n$ $(\sigma \to 0^+)$. This proves (1).

If we put

$$f_\sigma = |G_\sigma * h_{n,\sigma}|^{q-1} \operatorname{sgn}(G_\sigma * h_{n,\sigma}).$$

then

$$\left\{ \int_0^{2\pi} G_\sigma(x - y)\, f_\sigma(x)\, dx \right\}_{\sigma > 0}$$

is a bounded and equi-continuous subset of $C[0, 2\pi]$. Now (2) follows from (1) of Theorem 3.1 and $\lambda_{n,\sigma} \to \lambda_n \neq 0$. The proof is complete.

Denote by X_{2n} and $X_{2n,\sigma}$, respectively, the subspaces of the splines defined by G and G_σ with the simple knots $\{i\pi/n\}_{i=0}^{2n-1}$.

THEOREM 3.2. *Assume β_σ is the unique zero of $G_\sigma * h_{n,\sigma} \in [0, \pi/n)$. For $n > (2\mu(Q) - 1)\, \nu(Q)$,*

(1) *for any $f_\sigma = P + G_\sigma * h$, $P \in E(G)$, $h \in E^\perp(G)$, there exists an unique $S_{2n,\sigma}(f_\sigma) = S_{2n,\sigma}(f_\sigma, x) \in X_{2n,\sigma}$, which interpolates f_σ at $\{\beta_\sigma + (i\pi/n)\}_{i=0}^{2n-1}$;*

(2) $f_\sigma(x) - S_{2n,\sigma}(f_\sigma, x) = \int_0^{2\pi} \bar{M}_\sigma(x, y)\, dy := \bar{M}h$, *where*

$$\bar{M}_\sigma(x, y) = M_\sigma \begin{pmatrix} \beta_\sigma & \cdots & \beta_\sigma + \frac{(2n-1)\pi}{n} & x \\ 0 & \cdots & \frac{(2n-1)\pi}{n} & y \end{pmatrix} \Delta^{-1}$$

$$\Delta = M_\sigma \begin{pmatrix} \beta_\sigma & \cdots & \beta_\sigma + \frac{(2n-1)\pi}{n} \\ 0 & \cdots & \frac{(2n-1)\pi}{n} \end{pmatrix} (\neq 0);$$

(3) *there exists* $\varepsilon \in \{-1, 1\}$ *such that*

$$\bar{M}_\sigma(x, y) = \varepsilon \operatorname{sgn} \sin n(x - \beta_\sigma)\, |\bar{M}_\sigma(x, y)| \operatorname{sgn} \sin ny.$$

Proof. We first prove the unique existence of the interpolation spline. Equivalently, we prove that if $S_\sigma(x) = P(x) + \sum_{j=0}^{2n-1} c_j G_\sigma(x - (j\pi/n)) \in X_{2n \cdot \sigma}$ satisfies

$$S_\sigma\left(\beta_\sigma + \frac{j\pi}{n}\right) = 0, \qquad i = 0, 1, ..., 2n-1, \tag{3.6}$$

then $S_\sigma \equiv 0$.

In fact, if S_σ satisfies (3.6), and $S_\sigma \not\equiv 0$, then there is a constant c, such that $G_\sigma * h_{n,\sigma} - cS_\sigma$ has $(2n+1)$ distinct zeros. Therefore, $\operatorname{dis}(G_\sigma * h_{n,\sigma} - cS_\sigma) \leqslant \pi/n$, and for any positive integer $m \geqslant 2n+1$, $Z_m(G_\sigma * h_{n,\sigma}(\cdot) - cS_\sigma(\cdot)) \geqslant 2n+1$. By Lemma 2.1 (1) and Lemma 2.2, we have that

$$\begin{aligned} 2n+1 &\leqslant Z_m(G_\sigma * h_{n,\sigma}(\cdot) - cS_\sigma(\cdot)) \\ &\leqslant Z_{m - \deg Q}\left(\phi_\sigma * h_{n,\sigma}(\cdot) - c \sum_{j=0}^{2n-1} c_j \phi_\sigma\left(\cdot - \frac{j\pi}{n}\right)\right) \\ &= Z_{m - \deg Q}\left(\phi_{\sigma/2} * \left(\phi_{\sigma/2} * h_{n,\sigma}(\cdot) - c \sum_{j=0}^{2n-1} c_j \phi_{\sigma/2}\left(\cdot - \frac{j\pi}{n}\right)\right)\right) \\ &\leqslant S_c\left(\phi_{\sigma/2} * h_{n,\sigma}(\cdot) - c \sum_{j=0}^{2n-1} c_j \phi_{\sigma/2}\left(\cdot - \frac{j\pi}{n}\right)\right). \end{aligned}$$

On the other hand, for sufficiently small $\tau > 0$, we define a 2π-periodic function as follows:

$$\theta_\tau = \begin{cases} \frac{1}{\tau}, & |x| \leqslant \tau; \\ 0 & \tau < x < 2\pi - \tau. \end{cases}$$

Thus

$$S_c\left(\phi_{\sigma/2} * h_{n,\sigma}(\cdot) - c\sum_{j=0}^{2n-1} c_j\phi_{\sigma/2}\left(\cdot - \frac{j\pi}{n}\right)\right)$$
$$\leqslant \varliminf_{\tau\to 0^+} S_c\left(\phi_{\sigma/2} * \left(h_{n,\sigma}(\cdot) - c\sum_{j=0}^{2n-1} c_j\theta_\tau\left(\cdot - \frac{j\pi}{n}\right)\right)\right)$$
$$\leqslant \varliminf_{\tau\to 0^+} S_c\left(h_{n,\sigma}(\cdot) - c\sum_{j=0}^{2n-1} c_j\theta_\tau\left(\cdot - \frac{j\pi}{n}\right)\right)$$
$$\leqslant 2n.$$

The last inequality follows from the fact that $h_{n,\sigma}$ is continuous and $S_c(h_{n,\sigma}) = 2n$. This implies a contradiction. So we have proved that (1) holds and therefore $\Delta \neq 0$. By directly computing $\int_0^{2\pi} \bar{M}_\sigma(x, y)\, h(y)\, dy$, we obtain

$$\int_0^{2\pi} \bar{M}_\sigma(x, y)\, h(y)\, dy = f_\sigma(x) - s_\sigma(x),$$

where $s_\sigma \in X_{2n,\sigma}$ and $f_\sigma(\beta_\sigma + (i\pi/n)) = s_\sigma(\beta_\sigma + (i\pi/n))$, $i = 0, 1, ..., 2n-1$. Thus from the unicity of interpolation, (2) holds.

In what follows, we prove (3). Expanding $\bar{M}_\sigma$ by the last row we obtain that when $y \in [0, 2\pi)\backslash\{i\pi/n\}_{i=0}^{2n-1}$, $\bar{M}_\sigma \in X(\{i\pi/n\}_{i=0}^{2n-1} \cup \{y\}, \sigma)$ and $\bar{M}_\sigma \not\equiv 0$ (since the coefficient of $G_\sigma(x-y)$ is not zero). By Lemma 2.4, when $y \in ((i/n)\pi, ((i+1)/n)\pi)$, $\bar{M}_\sigma(\cdot, y)$ changes signs just at $\{\beta_\sigma - (i\pi/n)\}_{i=0}^{2n-1}$. Applying the same argument as above, it follows from the remark of Section 2 that when $x \in (\beta_\sigma + (i/n)\pi, \beta_\sigma + ((i+1)/n)\pi)$, $\bar{M}_\sigma(x, \cdot)$ changes signs just at $\{i\pi/n\}_{i=0}^{2n-1}$. This completes the proof of (3).

THEOREM 3.3. *Let $n > (2\mu(Q)-1)\,v(Q)$, β and h_n be given in Theorem 3.1. Then, for any $f \in W_p(Q(D))$, there exists a unique $S_{2n}(f) \in X_{2n}$ which interpolates f at $\{\beta + (i\pi/n)\}_{i=0}^{2n-1}$, and*

$$\sup\{\|f - S_{2n}(f)\|_q \mid f \in W_p(QD))\} = \lambda_n, \qquad 1 < q \leqslant p < \infty.$$

Proof. The procedure of the proof is similar to that of Proposition 2.7 in [13] or Theorem 2.2 in [2]. First, by Theorem 3.2, we can prove $\|f_\sigma - S_{2n}(f_\sigma)\|_q \leqslant \lambda_{n,\sigma}$, for any $f_\sigma = P + G_\sigma * h$, where $P \in E(G)$, $h \in E^\perp(G)$, $\|h\|_p \leqslant 1$ and $1 < q \leqslant p < \infty$. Second, by Lemma 3.2, we obtain a $S_{2n}(f) \in X_{2n}$ which interpolates f at $\{\beta + (i\pi/n)\}_{i=0}^{2n-1}$ and such that $\|f - S_{2n}(f)\|_q \leqslant \lambda_n$ for any $f \in W_p(Q(D))$. We omit the details. The uniqueness of such S_{2n} follows from the existence of that and $\dim X_{2n} = 2n$. Take $f = G * h_n$, then $S_{2n}(G * h_n) = 0$ and $\|G * h_n - S_{2n}(G * h_n)\|_q = \lambda_n$. This proves the theorem.

By Theorem 3.3, it follows that, for $n > (2\mu(Q)-1)\,\nu(Q)$ and $1 < q \leqslant p < \infty$, $\delta_{2n}(p, q) \leqslant \lambda_n$.

THEOREM 3.4. *Suppose* $n > (2\mu(Q)-1)\,\nu(Q)$ *and* $1 < q \leqslant p < \infty$. *Then*

$$E(W_p(Q(D)), X_{2n})_q := \sup_{f \in W_p(Q(D))} \inf_{S \in X_{2n}} \|f - S\|_q = \lambda_n(p, q, G).$$

Proof. Since $|G * h_n|^{q-1} \operatorname{sgn}(G * h_n)$ satisfies (3.1), then for $n > (2\mu(Q)-1)\,\nu(Q)$ and any $P(x) \in E(G)$ we obtain

$$\int_0^{2\pi} P(x)\,|(G * h_n)(x)|^{q-1} \operatorname{sgn}[(G * h_n)(x)]\,dx = 0.$$

By Theorem 3.1, we have

$$\int_0^{2\pi} G\left(x - \frac{i\pi}{n}\right) |(G * h_n)(x)|^{q-1} \operatorname{sgn}[(G * h_n)(x)]\,dx = 0.$$

Therefore zero is the best approximant from X_{2n} to $G * h_n$ in L_q. This means that

$$E(W_p(Q(D)), X_{2n})_q \geqslant \lambda_n.$$

The converse inequality is obtained by Theorem 3.3. The proof is complete.

4. THE MAIN RESULTS

THEOREM 4.1. *Assume* $n > N(Q) := 3\nu(Q)(2\mu(Q)-1)$. *Then for* $p \in (1, \infty)$

$$d_{2n}(p, p) = d^{2n}(p, p) = \delta_{2n}(p, p) = \lambda_n(p, p, G) \leqslant b_{2n+1}(p, p).$$

Moreover,

(1) X_{2n} *is optimal for* $d_{2n}(p, p)$.

(2) $L^{2n} := \{f \in W_p(Q(D)) \mid f(i\pi/n) = 0, i = 0, 1, ..., 2n-1\}$ *is optimal for* $d^{2n}(p, p)$.

(3) S_{2n} *is an optimal operator of rank-*$2n$ *for* $\delta_{2n}(p, p)$.

In the proof of Theorem 4.1, we will use the following. Suppose β and h_n are given as in Theorem 3.1. Define 2π-periodic functions as follows.

$$\varphi_n(x)=\begin{cases}(2n)^{1-(1/q)}\,\lambda_n^{1-q}\,|(G*h_n)(x)|^{q-1}, & x\in\left[\beta,\beta+\dfrac{\pi}{n}\right);\\ 0, & x\in[\beta,\beta+2\pi)\Big\backslash\left[\beta,\beta+\dfrac{\pi}{n}\right);\end{cases}$$

$$f_j(x)=\begin{cases}|h_n(x)|, & x\in\left[\dfrac{j}{n}\pi,\dfrac{j+1}{n}\pi\right);\\ 0, & x\in[0,2\pi)\Big\backslash\left[\dfrac{j}{n}\pi,\dfrac{j+1}{n}\pi\right).\end{cases}$$

$j=0,1,...,2n-1$. Put

$$M_{2n}=\left\{\sum_{i=0}^{2n-1}a_if_i\,\middle|\,\sum_{i=0}^{2n-1}a_i\,\delta\left(x-\frac{i\pi}{n}\right)\in E^{\perp}(G)\right\}.$$

Lemma 4.1. *Suppose $n>N(Q)$ and $1<p\le q<\infty$. Then for any $P(x)\in E(G)$ and $f\in M_{2n}$, the inequality*

$$\lambda_n\,\|f\|_p\le\left\|\left(\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)(P(x)+(G*f)(x))\,dx\right)_{k=0}^{2n-1}\right\|_{l_q}\tag{4.1}$$

holds. Here $\Delta_k=[\beta+(k\pi/n),\beta+((k+1)\pi/n))$, and $\|\cdot\|_{l_q}$ denote the l_q-norm in $\mathbf{R}^{2n}$.

Proof. We first notice the fact that if $f=\sum_{i=0}^{2n-1}a_if_i\in M_{2n}$, then $\|f\|_p=(2n)^{-1/p}\,\|(a_i)_{i=0}^{2n-1}\|_{l_p}$. Consider the following extremal problem:

$$\mu:=\min\left\{\|f\|_p^{-1}\cdot\left\|\left(\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)(P(x)+(G*f)(x))\,dx\right)_{k=0}^{2n-1}\right\|_{l_q}\right|$$
$$\left.P\in E(G),\,f\in M_{2n}\backslash\{0\}\right\}.\tag{4.2}$$

Obviously, the minimum is attained at some $\hat P\in E(G)$, $\hat f:=\sum_{j=0}^{2n-1}\hat a_jf_j\in M_{2n}$. We can normalize $\hat P$ and $\hat f$ so that $|\hat a_j|\le 1$, $j=0,1,...,2n-1$, and $\hat a_m=(-1)^m$ for some m. Since $(\hat P,\hat f)$ is a critical point for (4.2), it must satisfy the following conditions:

$$\sum_{k=0}^{2n-1}\left|\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)(\hat P+G*\hat f)(x)\right|^{q-1}\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)P(x)\,dx$$
$$\times\operatorname{sgn}\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)(\hat P+G*\hat f)(x)\,dx=0,\tag{4.3}$$

for any $P \in E(G)$, and

$$\sum_{k=0}^{2n-1} \left| \int_{\Delta_k} \varphi_n\left(x - \frac{k\pi}{n}\right) (\hat{P} + G * \hat{f})(x) \right|^{q-1} \int_{\Delta_k} \varphi_n\left(x - \frac{k\pi}{n}\right) (G * f_i)(x)\, dx$$
$$\times \operatorname{sgn} \int_{\Delta_k} \varphi_n\left(x - \frac{k\pi}{n}\right) (\hat{P} + G * \hat{f})(x)\, dx$$
$$= \frac{1}{2n} \mu^q \|\hat{f}\|_q^{q-p} \cdot |\hat{a}_i|^{p-1} \operatorname{sgn} \hat{a}_i, \qquad i = 0, 1, ..., 2n-1. \tag{4.4}$$

It is easy to prove that (4.3) and (4.4) are also valid if μ, $\hat{f}$, and $\hat{P}$ are replaced by λ_n, h_n, and 0, respectively. Set

$$a_k = \int_{\Delta_k} \varphi_n\left(x - \frac{k\pi}{n}\right) (G * h_n)(x)\, dx;$$
$$b_k = \int_{\Delta_k} \varphi_n\left(x - \frac{k\pi}{n}\right) (\hat{P} + G * \hat{f})(x)\, dx,$$
$$c_k = |a_k|^{q-1} \operatorname{sgn} a_k - |b_k|^{q-1} \operatorname{sgn} b_k,$$

$k = 0, 1, ..., 2n-1$, and

$$F(x) = \sum_{k=0}^{2n-1} c_k \varphi_n\left(x - \frac{k\pi}{n}\right).$$

From (4.3), (4.4), and the analogous formulas for h_n, λ_n, and 0, it follows that

$$\int_\beta^{\beta+2\pi} F(x)\, P(x)\, dx = 0, \qquad \forall P \in E(G). \tag{4.5}$$

$$\int_\beta^{\beta+2\pi} F(x)(G * f_i)(x)\, dx = \frac{1}{2n} [\lambda_n^q (-1)^i - \|\hat{f}\|_q^{q-p} \mu^q |\hat{a}_i|^{p-1} \operatorname{sgn} \hat{a}_i], \tag{4.6}$$

$i = 0, 1, ..., 2n-1$.

If $\lambda_n > \mu$, then from (4.6) and $\|\hat{f}\|^{q-p} \leqslant 1$, it follows that

$$2n = S_c^-\left(\left(\int_\beta^{\beta+2\pi} F(x)(G * f_i)(x)\, dx\right)_{j=0}^{2n-1}\right)$$
$$= S_c^-\left(\left(\int_{i\pi/n}^{(i+1)\pi/n} f_i(y)(G^* * F)(y)\, dy\right)_{j=0}^{2n-1}\right). \tag{4.7}$$

where $G^*(\cdot) := G(-\cdot)$, and $S_c^-(Y)$ denotes the cyclic variation of $Y \in \mathbf{R}^{2n}$

(cf. [12, p. 59]). By (4.7) and the non-negativeness of f_i, there exist $t_i \in (i\pi/n, ((i+1)\pi)/n)$ such that

$$[(G^* * F)(t_i)][(G^* * F)(t_{i+1})] < 0, \qquad i = 0, 1, ..., 2n-1,$$

where $t_{2n} = t_0 + 2\pi$. The above facts yield that $\operatorname{dis}(G^* * F) < 3\pi/n < \pi/(2\mu(Q)-1)\,\nu(Q)$. It follows from (4.5) that $F \in E^{\perp}(G) = E^{\perp}(G^*)$. From Lemma 2.1 and the remark in Section 2, we have

$$\begin{aligned} 2n &= S_c(G^* * F) \leqslant S_c(F) \\ &= S_c^-((c_i)_{i=0}^{2n-1}) = S_c^-((a_i - b_i)_{i=0}^{2n-1}) \\ &= S_c^-\left(\left(\int_{\Delta_i} \varphi_n\left(x - \frac{i\pi}{n}\right)[(G * h_n)(x) - \hat{P} - (G * \hat{f})(x)]\,dx\right)_{i=0}^{2n-1}\right) \end{aligned}$$

(Here we have used the relation $\operatorname{sgn}(a-b) = |a|^{q-1}\operatorname{sgn} a - |b|^{q-1}\operatorname{sgn} b$.)

Repeating the above procedure, we have $\operatorname{dis}(-\hat{P} + G * (h_n - \hat{f})) < \pi/(2\mu(Q)-1)\,\nu(Q)$. Therefore,

$$S_c(-\hat{P} + G * (h_n - \hat{f})) \leqslant S_c(h_n - \hat{f}).$$

From the non-negativeness of φ_n we obtain

$$\begin{aligned} 2n &= S_c^-\left(\left(\int_{\Delta_i} \varphi_n\left(x - \frac{i\pi}{n}\right)(-\hat{P} + G * (h_n - \hat{f})(x))\,dx\right)_{i=0}^{2n-1}\right) \\ &\leqslant S_c(-\hat{P} + G * (h_n - \hat{f})) \leqslant S_c(h_n - \hat{f}) \\ &= S_c\left(\sum_{i \neq m} ((-1)^i - \hat{a}_i)\,\hat{f}\right) \leqslant 2n - 2. \end{aligned}$$

This contradiction yields $\lambda_n \leqslant \mu$. On the other hand, by putting $P = 0$, $\hat{f} = h_n$ we get $\lambda_n \geqslant \mu$. Thus the lemma is proved.

By Lemma 4.1 and the method used in proving Lemma 3.4 of [2], we can prove

LEMMA 4.2. *If $n > N(Q)$ and $1 < p \leqslant q < \infty$, then*

$$d_{2n}(p, q) \geqslant \lambda_n.$$

LEMMA 4.3. *Assume $n > N(Q)$ and $1 < p \leqslant q < \infty$. Then*

$$d^{2n}(p, q) \geqslant \lambda_n(q', p', G^*),$$

where $(1/p) + (1/p') = (1/q) + (1/q') = 1$.

Proof. Let $U_{2n}=\text{span}\{u_i\}_{i=1}^{2n}\subseteq L_{q'}$. We are going to prove the following inequality:

$$\sup\{\|f\|_q \mid f\in W_p(Q(D)),\ f\perp U_{2n}\}\geqslant \lambda_n(q',p',G^*).$$

Obviously, we may suppose that the left hand side of the above inequality is finite. Therefore, if $P\in E(G)$ satisfies $P\perp U_{2n}$, then $P=0$, which means that the rank of the matrix $((u_i,g_j))_{i=1,j=1}^{2n,v}$ is v, where $(f,g)=\int_0^{2\pi} f(x)\,g(x)\,dx$, $v=\dim E(G)$ and $\{g_j\}_{j=1}^{v}$ is a basis of $E(G)$. Without loss of generality, we may suppose that

$$(u_i,P)=\sum_{j=1}^{v} a_{ij}(u_j,P),\qquad \forall P\in E(G),\qquad i=v+1,\ldots,2n.$$

Denote $v_i(y)=\int_0^{2\pi} G(x-y)\,u_i(x)\,dx$, and

$$w_j=g_j,\qquad j=1,\ldots,v,$$

$$w_j=v_j-\sum_{i=1}^{v} a_{ji}v_i,\qquad j=v+1,\ldots,2n.$$

It is easy to prove that

$$f(x)=P(x)+(G*h)(x)\perp U_{2n}$$

if and only if $h\perp W_{2n}:=\text{span}\{w_j\}_{j=1}^{2n}$. Therefore,

$$\begin{aligned}
&\sup\{\|f\|_q \mid f\in W_p(Q(D)),\ f\perp U_{2n}\}\\
&\quad=\sup\left\{\int_0^{2\pi} f(x)\,g(x)\,dx \mid f\in W_p(Q(D)),\ f\perp U_{2n},\ \|g\|_{q'}\leqslant 1\right\}\\
&\quad\geqslant\sup\left\{\int_0^{2\pi} h(y)\left(P(x)+\int_0^{2\pi} G(x-y)\,g(x)\,dx\right)dy \mid h\perp W_{2n},\right.\\
&\qquad\qquad\left.\|h\|_p\leqslant 1,\ P\in E(G^*),\ g\in E^{\perp}(G^*),\ \|g\|_{q'}\leqslant 1\right\}\\
&\quad\geqslant d_{2n}(W_{q'}(Q(-D)),L_{p'})\geqslant \lambda_n(q',p',G^*).
\end{aligned}$$

So we have proved the lemma.

Proof of Theorem 4.1. From Lemma 4.1 and

$$\left\|\left(\int_{\Delta_k}\varphi_n\left(x-\frac{k\pi}{n}\right)g(x)\,dx\right)_{k=0}^{2n-1}\right\|_{l_p}\leqslant \|g\|_p,$$

it follows that $b_{2n-1}(p, p) \geqslant \lambda_n$. By Theorem 3.3, Lemma 4.2 and the relations among the n-widths, we obtain

$$\begin{aligned}\lambda_n(p, p, G) \leqslant d_{2n}(p, p) \leqslant \varphi_{2n}(p, p) \leqslant \lambda_n(p, p, G),\\ \lambda_n(p', p', G^*) \leqslant d^{2n}(p, p) \leqslant \delta_{2n}(p, p).\end{aligned} \tag{4.8}$$

By Lemma 3.5 in [2], we get

$$\lambda_n(p', p', G^*) = \lambda_n(p, p, G).$$

Hence all the inequalities of (4.8) turn into equalities. Therefore, both (1) and (3) hold. Because of the explanations for (2) in [8] or [12], (2) also holds. Thus the proof of the theorem is complete.

COROLLARY 4.4. *If $n > N(Q)$, then*

$$d_{2n}(2, 2) = d^{2n}(2, 2) = \delta_{2n}(2, 2) = \frac{1}{|Q(in)|}, \qquad i = \sqrt{-1}.$$

Proof. We only need to show that for $n > N(Q)$

$$\lambda_n(2, 2, G) = \frac{1}{|Q(in)|}. \tag{4.9}$$

Notice that

$$|Q(iu)| = \prod_{j=1}^{r} \sqrt{u^2 + \lambda_j^2} \prod_{j=1}^{\mu(Q)} \sqrt{a_j^4 + 2a_j^2 b_j^2 + (u^2 - b_j^2)^2}$$

is a strictly increasing function of u for $u \in (\nu(Q), \infty)$.

By the definition of $D_{n,2}$, we know that for any $h \in D_{n,2}$ we have

$$h(t) = \sum_{k \in \mathbf{Z}, k \neq 0} c_{nk} e^{-inkt}, \qquad c_{nk} = \bar{c}_{-nk}.$$

Therefore (cf. [6, p. 456, (1.4)]),

$$(G * h)(t) = \sum_{k \in \mathbf{Z}, k \neq 0} \frac{c_{nk}}{Q(ink)} e^{-inkt}.$$

It follows from Parseval's formula that

$$\begin{aligned}\|G * h\|_2 &\leqslant \sqrt{2\pi} \left(\sum_{k \in \mathbf{Z}, k \neq 0} \left| \frac{c_{nk}}{Q(ink)} \right|^2 \right)^{1/2} \leqslant \frac{\sqrt{2\pi}}{|Q(ink)|} \left(\sum_{k \in \mathbf{Z}, k \neq 0} |c_{nk}|^2 \right)^{1/2}\\ &= \frac{\|h\|_2}{|Q(in)|} \leqslant \frac{1}{|Q(in)|}.\end{aligned}$$

On the other hand, if we take $h_n(x) = (1/\sqrt{\pi}) \sin nx \in D_{n,2}$, then $\|G * h_n\|_2 = 1/|Q(in)|$. Therefore (4.9) holds. The proof is complete.

Remark. It had been derived in [10] that $d_{2n}(2, 2) = |Q(in)|^{-1}$ for sufficiently large n by a method different from that in this paper.

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